

CONTENTS

List of Figures	xv
List of Tables	xxiii
Preface	xxv
Acknowledgments	xxvii
1 Partial Differential Equations	1
1.1 Selected general properties	2
1.1.1 Classification and examples	2
1.1.2 Hadamard's well-posedness	5
1.1.3 General existence and uniqueness results	9
1.1.4 Exercises	11
1.2 Second-order elliptic problems	13
1.2.1 Weak formulation of a model problem	13
1.2.2 Bilinear forms, energy norm, and energetic inner product	16
1.2.3 The Lax–Milgram lemma	18
1.2.4 Unique solvability of the model problem	18
1.2.5 Nonhomogeneous Dirichlet boundary conditions	20
1.2.6 Neumann boundary conditions	21
1.2.7 Newton (Robin) boundary conditions	22
1.2.8 Combining essential and natural boundary conditions	23
	vii

1.2.9	Energy of elliptic problems	24
1.2.10	Maximum principles and well-posedness	26
1.2.11	Exercises	29
1.3	Second-order parabolic problems	30
1.3.1	Initial and boundary conditions	30
1.3.2	Weak formulation	30
1.3.3	Existence and uniqueness of solution	31
1.3.4	Exercises	32
1.4	Second-order hyperbolic problems	33
1.4.1	Initial and boundary conditions	33
1.4.2	Weak formulation and unique solvability	34
1.4.3	The wave equation	34
1.4.4	Exercises	35
1.5	First-order hyperbolic problems	36
1.5.1	Conservation laws	36
1.5.2	Characteristics	38
1.5.3	Exact solution to linear first-order systems	39
1.5.4	Riemann problem	41
1.5.5	Nonlinear flux and shock formation	43
1.5.6	Exercises	44
2	Continuous Elements for 1D Problems	45
2.1	The general framework	45
2.1.1	The Galerkin method	46
2.1.2	Orthogonality of error and Céa's lemma	49
2.1.3	Convergence of the Galerkin method	50
2.1.4	Ritz method for symmetric problems	51
2.1.5	Exercises	51
2.2	Lowest-order elements	52
2.2.1	Model problem	52
2.2.2	Finite-dimensional subspace $V_n \subset V$	52
2.2.3	Piecewise-affine basis functions	53
2.2.4	The system of linear algebraic equations	54
2.2.5	Element-by-element assembling procedure	56
2.2.6	Refinement and convergence	57
2.2.7	Exercises	57
2.3	Higher-order numerical quadrature	59
2.3.1	Gaussian quadrature rules	60
2.3.2	Selected quadrature constants	62
2.3.3	Adaptive quadrature	63
2.3.4	Exercises	65
2.4	Higher-order elements	66

2.4.1	Motivation problem	66
2.4.2	Affine concept: reference domain and reference maps	67
2.4.3	Transformation of weak forms to the reference domain	69
2.4.4	Higher-order Lagrange nodal shape functions	70
2.4.5	Chebyshev and Gauss–Lobatto nodal points	71
2.4.6	Higher-order Lobatto hierarchic shape functions	74
2.4.7	Constructing basis of the space $V_{h,p}$	76
2.4.8	Data structures	77
2.4.9	Assembling algorithm	79
2.4.10	Exercises	82
2.5	The sparse stiffness matrix	84
2.5.1	Compressed sparse row (CSR) data format	84
2.5.2	Condition number	84
2.5.3	Conditioning of shape functions	86
2.5.4	Stiffness matrix for the Lobatto shape functions	88
2.5.5	Exercises	89
2.6	Implementing nonhomogeneous boundary conditions	89
2.6.1	Dirichlet boundary conditions	89
2.6.2	Combination of essential and natural conditions	91
2.6.3	Exercises	92
2.7	Interpolation on finite elements	93
2.7.1	The Hilbert space setting	93
2.7.2	Best interpolant	94
2.7.3	Projection-based interpolant	96
2.7.4	Nodal interpolant	99
2.7.5	Exercises	102
3	General Concept of Nodal Elements	103
3.1	The nodal finite element	103
3.1.1	Unisolvency and nodal basis	104
3.1.2	Checking unisolvency	106
3.2	Example: lowest-order Q^1 - and P^1 -elements	107
3.2.1	Q^1 -element	108
3.2.2	P^1 -element	110
3.2.3	Invertibility of the quadrilateral reference map \boldsymbol{x}_K	113
3.3	Interpolation on nodal elements	114
3.3.1	Local nodal interpolant	115
3.3.2	Global interpolant and conformity	116
3.3.3	Conformity to the Sobolev space H^1	119
3.4	Equivalence of nodal elements	120
3.5	Exercises	122

4	Continuous Elements for 2D Problems	125
4.1	Lowest-order elements	126
4.1.1	Model problem and its weak formulation	126
4.1.2	Approximations and variational crimes	127
4.1.3	Basis of the space $V_{h,p}$	129
4.1.4	Transformation of weak forms to the reference domain	131
4.1.5	Simplified evaluation of stiffness integrals	133
4.1.6	Connectivity arrays	134
4.1.7	Assembling algorithm for Q^1/P^1 -elements	135
4.1.8	Lagrange interpolation on Q^1/P^1 -meshes	137
4.1.9	Exercises	137
4.2	Higher-order numerical quadrature in 2D	139
4.2.1	Gaussian quadrature on quads	139
4.2.2	Gaussian quadrature on triangles	139
4.3	Higher-order nodal elements	142
4.3.1	Product Gauss–Lobatto points	142
4.3.2	Lagrange–Gauss–Lobatto $Q^{p,r}$ -elements	143
4.3.3	Lagrange interpolation and the Lebesgue constant	148
4.3.4	The Fekete points	149
4.3.5	Lagrange–Fekete P^p -elements	152
4.3.6	Basis of the space $V_{h,p}$	154
4.3.7	Data structures	157
4.3.8	Connectivity arrays	160
4.3.9	Assembling algorithm for Q^p/P^p -elements	162
4.3.10	Lagrange interpolation on Q^p/P^p -meshes	166
4.3.11	Exercises	166
5	Transient Problems and ODE Solvers	167
5.1	Method of lines	168
5.1.1	Model problem	168
5.1.2	Weak formulation	168
5.1.3	The ODE system	169
5.1.4	Construction of the initial vector	170
5.1.5	Autonomous systems and phase flow	171
5.2	Selected time integration schemes	172
5.2.1	One-step methods, consistency and convergence	173
5.2.2	Explicit and implicit Euler methods	175
5.2.3	Stiffness	177
5.2.4	Explicit higher-order RK schemes	179
5.2.5	Embedded RK methods and adaptivity	182
5.2.6	General (implicit) RK schemes	184

5.3	Introduction to stability	185
5.3.1	Autonomization of RK methods	186
5.3.2	Stability of linear autonomous systems	187
5.3.3	Stability functions and stability domains	188
5.3.4	Stability functions for general RK methods	191
5.3.5	Maximum consistency order of IRK methods	193
5.3.6	A -stability and L -stability	194
5.4	Higher-order IRK methods	197
5.4.1	Collocation methods	197
5.4.2	Gauss and Radau IRK methods	200
5.4.3	Solution of nonlinear systems	202
5.5	Exercises	205
6	Beam and Plate Bending Problems	209
6.1	Bending of elastic beams	210
6.1.1	Euler–Bernoulli model	210
6.1.2	Boundary conditions	212
6.1.3	Weak formulation	214
6.1.4	Existence and uniqueness of solution	214
6.2	Lowest-order Hermite elements in 1D	216
6.2.1	Model problem	216
6.2.2	Cubic Hermite elements	218
6.3	Higher-order Hermite elements in 1D	220
6.3.1	Nodal higher-order elements	220
6.3.2	Hierarchic higher-order elements	222
6.3.3	Conditioning of shape functions	225
6.3.4	Basis of the space $V_{h,p}$	226
6.3.5	Transformation of weak forms to the reference domain	228
6.3.6	Connectivity arrays	228
6.3.7	Assembling algorithm	231
6.3.8	Interpolation on Hermite elements	233
6.4	Hermite elements in 2D	236
6.4.1	Lowest-order elements	236
6.4.2	Higher-order Hermite–Fekete elements	238
6.4.3	Design of basis functions	240
6.4.4	Global nodal interpolant and conformity	242
6.5	Bending of elastic plates	242
6.5.1	Reissner–Mindlin (thick) plate model	243
6.5.2	Kirchhoff (thin) plate model	246
6.5.3	Boundary conditions	248
6.5.4	Weak formulation and unique solvability	250
6.5.5	Babuška’s paradox of thin plates	254

6.6	Discretization by H^2 -conforming elements	255
6.6.1	Lowest-order (quintic) Argyris element, unisolvency	255
6.6.2	Local interpolant, conformity	256
6.6.3	Nodal shape functions on the reference domain	257
6.6.4	Transformation to reference domains	259
6.6.5	Design of basis functions	260
6.6.6	Higher-order nodal Argyris–Fekete elements	265
6.7	Exercises	266
7	Equations of Electromagnetics	269
7.1	Electromagnetic field and its basic characteristics	270
7.1.1	Integration along smooth curves	270
7.1.2	Maxwell’s equations in integral form	272
7.1.3	Maxwell’s equations in differential form	273
7.1.4	Constitutive relations and the equation of continuity	274
7.1.5	Media and their characteristics	275
7.1.6	Conductors and dielectrics	275
7.1.7	Magnetic materials	276
7.1.8	Conditions on interfaces	277
7.2	Potentials	279
7.2.1	Scalar electric potential	279
7.2.2	Scalar magnetic potential	281
7.2.3	Vector potential and gauge transformations	281
7.2.4	Potential formulation of Maxwell’s equations	283
7.2.5	Other wave equations	283
7.3	Equations for the field vectors	284
7.3.1	Equation for the electric field	285
7.3.2	Equation for the magnetic field	285
7.3.3	Interface and boundary conditions	286
7.3.4	Time-harmonic Maxwell’s equations	287
7.3.5	Helmholtz equation	288
7.4	Time-harmonic Maxwell’s equations	289
7.4.1	Normalization	289
7.4.2	Model problem	290
7.4.3	Weak formulation	290
7.4.4	Existence and uniqueness of solution	293
7.5	Edge elements	300
7.5.1	Conformity requirements of the space $\mathbf{H}(\text{curl})$	301
7.5.2	Lowest-order (Whitney) edge elements	302
7.5.3	Higher-order edge elements of Nédélec	309
7.5.4	Transformation of weak forms to the reference domain	314
7.5.5	Interpolation on edge elements	316

7.5.6	Conformity of edge elements to the space $\mathbf{H}(\text{curl})$	317
7.6	Exercises	318
Appendix A: Basics of Functional Analysis		319
A.1	Linear spaces	320
A.1.1	Real and complex linear space	320
A.1.2	Checking whether a set is a linear space	321
A.1.3	Intersection and union of subspaces	323
A.1.4	Linear combination and linear span	326
A.1.5	Sum and direct sum of subspaces	327
A.1.6	Linear independence, basis, and dimension	328
A.1.7	Linear operator, null space, range	332
A.1.8	Composed operators and change of basis	337
A.1.9	Determinants, eigenvalues, and eigenvectors	339
A.1.10	Hermitean, symmetric, and diagonalizable matrices	341
A.1.11	Linear forms, dual space, and dual basis	343
A.1.12	Exercises	345
A.2	Normed spaces	348
A.2.1	Norm and seminorm	348
A.2.2	Convergence and limit	352
A.2.3	Open and closed sets	355
A.2.4	Continuity of operators	357
A.2.5	Operator norm and $\mathcal{L}(U, V)$ as a normed space	361
A.2.6	Equivalence of norms	363
A.2.7	Banach spaces	366
A.2.8	Banach fixed point theorem	371
A.2.9	Lebesgue integral and L^p -spaces	375
A.2.10	Basic inequalities in L^p -spaces	380
A.2.11	Density of smooth functions in L^p -spaces	384
A.2.12	Exercises	386
A.3	Inner product spaces	389
A.3.1	Inner product	389
A.3.2	Hilbert spaces	394
A.3.3	Generalized angle and orthogonality	395
A.3.4	Generalized Fourier series	399
A.3.5	Projections and orthogonal projections	401
A.3.6	Representation of linear forms (Riesz)	405
A.3.7	Compactness, compact operators, and the Fredholm alternative	407
A.3.8	Weak convergence	408
A.3.9	Exercises	409
A.4	Sobolev spaces	412
A.4.1	Domain boundary and its regularity	412

A.4.2	Distributions and weak derivatives	414
A.4.3	Spaces $W^{k,p}$ and H^k	418
A.4.4	Discontinuity of H^1 -functions in \mathbb{R}^d , $d \geq 2$	420
A.4.5	Poincaré–Friedrichs’ inequality	421
A.4.6	Embeddings of Sobolev spaces	422
A.4.7	Traces of $W^{k,p}$ -functions	424
A.4.8	Generalized integration by parts formulae	425
A.4.9	Exercises	426
Appendix B: Software and Examples		427
B.1	Sparse Matrix Solvers	427
B.1.1	The sMatrix utility	428
B.1.2	An example application	430
B.1.3	Interfacing with PETSc	433
B.1.4	Interfacing with Trilinos	436
B.1.5	Interfacing with UMFPACK	439
B.2	The High-Performance Modular Finite Element System HERMES	439
B.2.1	Modular structure of HERMES	440
B.2.2	The elliptic module	441
B.2.3	The Maxwell’s module	442
B.2.4	Example 1: L-shape domain problem	444
B.2.5	Example 2: Insulator problem	448
B.2.6	Example 3: Sphere-cone problem	451
B.2.7	Example 4: Electrostatic micromotor problem	455
B.2.8	Example 5: Diffraction problem	458
	References	461
	Index	468